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Functions and Derivations of C^*-Algebras

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It is shown that there is a closed symmetric derivation δ of a C^*-algebra with dense domain $\mathscr{D}(\delta)$, an element $A = A^* \in \mathscr{D}(\delta)$, and a C^1-function f such that $f(A) \notin \mathscr{D}(\delta)$. Some estimates are derived for $\| \delta(|A|)\|$ and $\| \delta(A_+^\alpha)\|$, where $0 < \alpha < 1$. It is shown that there exists a family of one–one self-adjoint operators $S(t)$ in $\mathscr{L}(H)$ which depends linearly on t, while $|S(t)|$ is not differentiable. It is also shown that there exists $T \in \mathscr{L}(H)$ which is not C^1-self-adjoint even though it satisfies $\| \exp(itT)\| \leqslant C(1 + |t|)$ for all $t \in \mathbb{R}$.

1. INTRODUCTION

If δ is a closed symmetric derivation[1] of a C^*-algebra with dense domain $\mathscr{D}(\delta)$, what conditions on a complex-valued function f will ensure that $f(A) \in \mathscr{D}(\delta)$ for all $A = A^* \in \mathscr{D}(\delta)$? Bratteli and Robinson [2] showed that a sufficient condition is $\int |p\hat{f}(p)|\, dp < \infty$, and indeed that $\| \delta(f(A))\| \leqslant \int |p\hat{f}(p)|\, dp \,\| \delta(A)\|$ for all $A = A^* \in \mathscr{D}(\delta)$. (Here $\hat{f}$ denotes the Fourier transform of f.) A stronger result, namely, $\| \delta(f(A))\| \leqslant \sup |f'| \,\| \delta(A)\|$ for all $f \in C^1$ and all $A = A^* \in \mathscr{D}(\delta)$, appears in a paper of Powers [13], but, as was noted by Bratteli and Robinson, the proof is erroneous. In fact this result is not true in general, as is shown in Section 2 using the function $f(x) = |x|$, suitably smoothed at $x = 0$. Indeed a somewhat weaker conjecture of Chi [3], namely that $f(A) \in \mathscr{D}(\delta)$ whenever $f \in C^1$ and $A = A^* \in \mathscr{D}(\delta)$, is also shown not to hold.

Some estimates which do hold for $\| \delta(|A|)\|$ are obtained in Section 3, together with estimates for $\| \delta(A_+^\alpha)\|$ where $0 < \alpha < 1$ and $A_+ = \frac{1}{2}(A + |A|)$. Two methods of proof are used in order to indicate different techniques that are available for obtaining such estimates.

Section 4 contains results about the smoothness of the map $A \rightsquigarrow |A|$. In particular, a question of Kato is answered by showing that there are bounded self-adjoint operators S_1 and S_2 in $\mathscr{L}(H)$ such that, for all $t \in \mathbb{R}$, $S_1 + tS_2$ is

[1] See [1, 2] for definitions. Only C^*-algebras having identity elements will be considered.

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one–one, and $|S_1 + tS_2|$ is not a differentiable function of t at $t = 0$. The example, which is a modification of an example of Kato [8], is constructed using the results of Section 2.

Chi noted that his conjecture would follow if a conjecture of Kantorovitz on C^m-self-adjoint operators were valid. Consequently the results of Section 2 can be used to construct a counterexample to Kantorovitz's conjecture. This is done in Section 5.

2. Counterexamples

In a previous paper [10] it was shown that for each positive integer n there exist self-adjoint operators A and B in a finite-dimensional Hilbert space H such that $\| |A| B - B |A| \| > n \| AB - BA \|$ and $0 \notin \sigma(A)$. On defining f to be a C^1-function with $\sup |f'| \leqslant 1$ satisfying $f(x) = |x|$ for all $x \in \sigma(A)$, and defining the derivation δ of $\mathscr{L}(H)$ by $\delta(A) = i(AB - BA)$, we have that $\| \delta(f(A))\| > n \| \delta(A)\|$, in contradiction to Powers' statement. However, to find a counterexample to Chi's conjecture, more substantial modifications are required.

The original construction used in [10] has been simplified by Kahan, and his ideas will be incorporated here. The papers [5, 6] of Kahan contain some interesting related results.

For $\alpha \geqslant 0$, denote by f_α the function on $[-e^{-1}, e^{-1}]$ defined by

$$\begin{aligned} f_\alpha(x) &= |x| (\log |\log | e^{-1} |x||)^{-\alpha}, && 0 < |x| \leqslant e^{-1}, \\ &= 0, && x = 0. \end{aligned}$$

Note that $f_0(x) = |x|$, and that f_α is of class C^1 when $\alpha > 0$.

Theorem 1. *For every integer $m \geqslant 3$ there is a self-adjoint operator U and a skew-adjoint operator V in the Hilbert space $\mathbb{C}^m$ satisfying $e^{-m} \leqslant U \leqslant e^{-1}$, $\| UV + VU \| \leqslant \pi$, and $\| f_\alpha(U)V - Vf_\alpha(U)\| > \frac{1}{8}(\log(\frac{1}{2}m))^{1-\alpha}$ for $0 \leqslant \alpha < 1$.*

Proof. Let W be the skew-adjoint operator with matrix (W_{ij}), where $W_{ij} = (j - i)^{-1}$ if $i \neq j$, $W_{ii} = 0$, and the indices range from 1 to m. (W_{ij}) is the $m \times m$ Toeplitz matrix corresponding to the function $g(\theta) = (-1)^{1/2}(\pi - \theta)$ on $0 < \theta < 2\pi$, so $\| W \| \leqslant \pi$ (cf. [5]). Let U denote the self-adjoint operator with diagonal matrix given by $U_{ii} = u_i = e^{-i}$. Let V be the skew-adjoint operator with matrix elements $V_{ij} = W_{ij}(u_i + u_j)^{-1}$. Then $(UV + VU)_{ij} = W_{ij}$, so $\| UV + VU \| \leqslant \pi$. Now let $S = f_\alpha(U)V - Vf_\alpha(U)$. Then

$$S_{ij} = (f_\alpha(u_i) - f_\alpha(u_j))\, W_{ij}(u_i + u_j)^{-1} \geqslant 0.$$

If $i < j$,

$$\begin{aligned} S_{ij} &= (f_\alpha(e^{-i}) - f_\alpha(e^{-j}))(j-i)^{-1}(e^{-i}+e^{-j})^{-1} \\ &= \left(\frac{e^{-i}}{(\log(i+1))^\alpha} - \frac{e^{-j}}{(\log(j+1))^\alpha}\right)\Big/(j-i)(e^{-i}+e^{-j}) \\ &\geqslant \frac{e^{-i}-e^{-j}}{2(j-i)\,e^{-i}(\log(j+1))^\alpha} \\ &\geqslant \frac{1-e^{-1}}{2(j-i)(\log(j+1))^\alpha}. \end{aligned}$$

Hence, if $2 \leqslant j \leqslant m$,

$$\begin{aligned} \sum_{i=1}^m S_{ij} \geqslant \sum_{i=1}^{j-1} S_{ij} &\geqslant \tfrac{1}{2}(1-e^{-1})(\log(j+1))^{-\alpha}\left\{\frac{1}{j-1} + \frac{1}{j-2} + \cdots + \frac{1}{2} + 1\right\} \\ &> \tfrac{1}{2}(1-e^{-1})(\log(j+1))^{-\alpha}\log j \\ &\geqslant \tfrac{1}{2}(1-e^{-1})(\log 2)(\log 3)^{-1}(\log(j+1))^{1-\alpha} \\ &> 26^{-1/2}(\log(j+1))^{1-\alpha}. \end{aligned}$$

Let $u = (1, 1,\ldots, 1)$. Then $\| u \|^2 = m$, so

$$\begin{aligned} 26m \, \| S \|^2 \geqslant 26 \, \| uS \|^2 &= 26 \sum_{j=1}^m \left\{\sum_{i=1}^m S_{ij}\right\}^2 \\ &> \sum_{j=2}^m (\log(j+1))^{2-2\alpha} \\ &> \sum_{j=[(m+1)/2]}^m (\log(\tfrac{1}{2}m))^{2-2\alpha} \\ &> (\tfrac{1}{2}m)(\log(\tfrac{1}{2}m))^{2-2\alpha}. \end{aligned}$$

We conclude that $\| f_\alpha(U)V - Vf_\alpha(U) \| = \| S \| > \frac{1}{8}(\log(\frac{1}{2}m))^{1-\alpha}$. ∎

THEOREM 2. *For every integer $m \geqslant 3$ there exist self-adjoint operators A_m and B_m in the Hilbert space $\mathbb{C}^{2m}$ satisfying $e^{-m} \leqslant | A_m | \leqslant e^{-1}$, $\| A_m B_m - B_m A_m \| \leqslant \pi$, and $\| f_\alpha(A_m)B_m - B_m f_\alpha(A_m) \| > \frac{1}{8}(\log(\frac{1}{2}m))^{1-\alpha}$ for $0 \leqslant \alpha < 1$.*

Proof. For operators U and V defined as above, let

$$A_m = \begin{bmatrix} U & 0 \\ 0 & -U \end{bmatrix} \quad \text{and} \quad B_m = \begin{bmatrix} 0 & V \\ V^* & 0 \end{bmatrix}.$$

$$\therefore A_m B_m - B_m A_m = \begin{bmatrix} 0 & UV + VU \\ -(UV+VU)^* & 0 \end{bmatrix}$$

so

$$\| A_m B_m - B_m A_m \| \leqslant \pi.$$

Also

$$f_\alpha(A_m) = \begin{bmatrix} f_\alpha(U) & 0 \\ 0 & f_\alpha(U) \end{bmatrix},$$

so

$$f_\alpha(A_m)B_m - B_m f_\alpha(A_m) = \begin{bmatrix} 0 & f_\alpha(U)V - Vf_\alpha(U) \\ -(f_\alpha(U)V - Vf_\alpha(U))^* & 0 \end{bmatrix}.$$

$$\therefore \| f_\alpha(A_m)B_m - B_m f_\alpha(A_m) \| > \frac{1}{8}(\log(\tfrac{1}{2}m))^{1-\alpha}. \quad \blacksquare$$

THEOREM 3. *There are self-adjoint operators A and B in a separable Hilbert space H such that $\| A \| \leqslant e^{-1}$, $A(\mathscr{D}(B)) \subset \mathscr{D}(B)$, $\| AB - BA \| \leqslant \pi$ and $f_\alpha(A)B - Bf_\alpha(A)$ is unbounded for $0 \leqslant \alpha < 1$.*

Proof. Let $H = \bigoplus_{m \geqslant 3} \mathbb{C}^{2m}$, $A = \bigoplus A_m$ and $B = \bigoplus B_m$, where A_m and B_m are defined as above. Then A and B have the required properties. (To check that $A(\mathscr{D}(B)) \subset \mathscr{D}(B)$, note that $AB - BA$ is defined and bounded on the set $\{u = \bigoplus u_m \mid \text{all but finitely many of the } u_m \text{ are zero}\}$, which is a core of B.) $\blacksquare$

THEOREM 4. *There is a closed symmetric derivation δ of a C^*-algebra $\mathfrak{a}$ with dense domain $\mathscr{D}(\delta)$, an element $A = A^* \in \mathscr{D}(\delta)$, and a C^1-function f defined on a closed interval containing the spectrum of A, such that $f(A) \notin \mathscr{D}(\delta)$.*

Proof. Choose A, B and H as in theorem 3, and, for each $t \in \mathbb{R}$, define $\tau_t : \mathscr{L}(H) \to \mathscr{L}(H)$ by $\tau_t(C) = \exp(itB)C\exp(-itB)$. Let $\mathfrak{a}$ be the C^*-algebra generated by $\{\tau_t(A) \mid t \in \mathbb{R}\}$. Then τ_t is a strongly continuous one-parameter group of $*$-automorphisms of $\mathfrak{a}$. Let δ be its infinitesimal generator. Then δ is a closed symmetric derivation of $\mathfrak{a}$ with dense domain $\mathscr{D}(\delta) = \{C \in \mathfrak{a} \mid C(\mathscr{D}(B)) \subset \mathscr{D}(B) \text{ and } \| BC - CB \| < \infty\}$, and $\delta(C) = i(BC - CB)$ for all $C \in \mathscr{D}(\delta)$. On taking f to be one of the functions f_α defined above with $\alpha \in (0, 1)$, we have that $A \in \mathscr{D}(\delta)$ and $f(A) \notin \mathscr{D}(\delta)$. $\blacksquare$

Note that we cannot prove the theorem by simply defining δ in $\mathscr{L}(H)$ by $\delta(C) = i(BC - CB)$, with the natural domain, for such a domain is not norm-dense.

Theorem 4 provides a counterexample to the conjecture of Chi. Note that theorem 4 cannot be strengthened by requiring that f be a C^2-function, for such an f can be modified outside $\sigma(A)$ so that the modified function satisfies $\int | p\hat{f}(p)| < \infty$. More particularly, if g_α is a function with compact support which is C^2 except at zero and satisfies $g_\alpha(x) = | \log | e^{-1}x | |^{-1} f_\alpha(x)$ in a neighborhood of zero, then it can be shown using Theorem 9.3 of [14] that $\int | p\hat{g}_\alpha(p)| < \infty$ if $\alpha > 1$ and $\int | p\hat{g}_\alpha(p)| = \infty$ if $0 < \alpha \leqslant 1$.

3. Estimates

A general principle concerning counterexamples based on the result of [10] quoted at the start of Section 2 seems to be the following. If a result has been disproved by using such a counterexample, then a suitable modification of the result by terms of the order of $(\log\log)^\alpha$, with $0 < \alpha < 1$, will also be invalid. If however it is suitably modified by adding logarithmic terms or by taking fractional powers, then it may become valid. This principle is illustrated in the following theorems, in which estimates which do hold for $\delta(|A|)$ and $\delta(A_+{}^\alpha)$ are derived. Here $A_+ = \frac{1}{2}(A + |A|)$. These theorems build on results about commutators due to Nirenberg and Treves [12, Lemma 3.1], Bruce Evans, and Tosio Kato. It will not be assumed that δ is symmetric or densely defined unless stated otherwise.

Theorem 5. *Let δ be a closed derivation of a C^*-algebra with $I \in \mathscr{D}(\delta)$, and let A be an element such that both A and A^* belong to $\mathscr{D}(\delta)$.*

(i) *If A is invertible then $|A| \in \mathscr{D}(\delta)$ and*

$$\|\delta(|A|)\| \leqslant \pi^{-1}(\|\delta(A)\| + \|\delta(A^*)\|)(2\tfrac{1}{2} + \log(\|A\|\,\|A^{-1}\|)).$$

(ii) *If δ is bounded then*

$$\|\delta(|A|)\| \leqslant \pi^{-1}(\|\delta(A)\| + \|\delta(A^*)\|)(4 + \log\{\|\delta\|\,\|A\|\,(\|\delta(A)\| + \|\delta(A^*)\|)^{-1}\}).$$

Proof. Let $0 < a \leqslant b = \|A\|^2$, where a will be chosen below, and use the formula which appears in [9, p. 282],

$$|A| = \pi^{-1}\int_0^\infty t^{-1/2}A^*A(t + A^*A)^{-1}\,dt.$$

$$\therefore \|\delta(|A|)\| \leqslant \pi^{-1}\left\|\delta\left(\int_0^a t^{-1/2}A^*A(t + A^*A)^{-1}\,dt\right)\right\| + \pi^{-1}\left\|\delta\left(\int_a^\infty t^{-1/2}A^*A(t + A^*A)^{-1}\,dt\right)\right\|.$$

On justifying questions about domains as in [1, Theorems 2 and 3], and using the inequalities $\|(t + A^*A)^{-1}\| \leqslant \min(t^{-1}, \|\,|A|^{-2}\|) = \min(t^{-1}, \|A^{-1}\|^2)$ and $\|A(t + A^*A)^{-1}\| = \|\,|A|\,(t + |A|^2)^{-1}\| \leqslant \min(\frac{1}{2}t^{-1/2}, t^{-1}\|A\|)$, we find that

$$\left\|\delta\left(\int_a^\infty t^{-1/2}A^*A(t + A^*A)^{-1}\,dt\right)\right\|$$

$$= \left\|\int_a^\infty t^{1/2}(t + A^*A)^{-1}(A^*\delta(A) + \delta(A^*)A)(t + A^*A)^{-1}\,dt\right\|$$

$$\leqslant \int_a^\infty t^{1/2} \min(\tfrac{1}{2}t^{-1/2}, t^{-1} \| A \|)(\| \delta(A)\| + \| \delta(A^*)\|)t^{-1}\, dt$$

$$\leqslant (\| \delta(A)\| + \| \delta(A^*)\|) \left\{\int_a^b \tfrac{1}{2}t^{-1}\, dt + \int_b^\infty t^{-3/2} \| A \|\, dt\right\}$$

$$= (\| \delta(A)\| + \| \delta(A^*)\|)(\log(a^{-1/2} \| A \|) + 2).$$

We now consider parts (i) and (ii) separately.

(i) $$\left\| \delta \left(\int_0^a t^{-1/2}A^*A(t + A^*A)^{-1}\, dt\right)\right\|$$

$$= \left\| \int_0^a t^{1/2}(t + A^*A)^{-1}(A^*\delta(A) + \delta(A^*)A)(t + A^*A)^{-1}\, dt \right\|$$

$$\leqslant \int_0^a \tfrac{1}{2}(\| \delta(A)\| + \| \delta(A^*)\|) \| A^{-1} \|^2\, dt$$

$$= (\| \delta(A)\| + \| \delta(A^*)\|)(\tfrac{1}{2}a \| A^{-1} \|^2).$$

On choosing $a = \| A^{-1} \|^{-2}$, we conclude that

$$\| \delta(| A |)\| \leqslant \pi^{-1}(\| \delta(A)\| + \| \delta(A^*)\|)(2\tfrac{1}{2} + \log(\| A \| \| A^{-1} \|)).$$

(ii) $$\left\| \delta \left(\int_0^a t^{-1/2}A^*A(t + A^*A)^{-1}\, dt\right)\right\| \leqslant \| \delta \| \int_0^a t^{-1/2}\, dt$$

$$= 2 \| \delta \| a^{1/2}.$$

On choosing $a = (\| \delta(A)\| + \| \delta(A^*)\|)^2 \| \delta \|^{-2}$, we conclude that

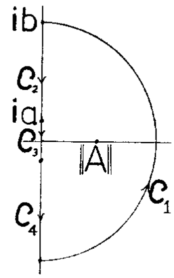
$$\| \delta(| A |)\| \leqslant \pi^{-1}(\| \delta(A)\| + \| \delta(A^*)\|)(4 + \log\{\| \delta \| \| A \| (\| \delta(A)\| + \| \delta(A^*)\|)^{-1}\}). \quad \blacksquare$$

It is a consequence of Theorem 3, with $\alpha = 0$, that the "log" terms cannot be deleted. It was shown by Bruce Evans that they cannot be improved to "$(\log\log)^\beta$" with $0 < \beta < 1$.

THEOREM 6. *If δ is a bounded derivation of a C^*-algebra $\mathfrak{a}$ with $\mathscr{D}(\delta) = \mathfrak{a}$, A is a self-adjoint element, and* $0 < \alpha < 1$, *then*

$$\| \delta(A_+^\alpha)\| \leqslant 2(1 - \alpha)^{-1} \| \delta(A)\|^\alpha \| \delta \|^{1-\alpha}.$$

Proof. Let $\mathscr{C}$ be the closed curve shown, where $b > 2 \| A \|$ and $a = \| \delta(A)\| \| \delta \|^{-1}$. Then $A_+^\alpha = (2\pi)^{-1} \int_{\mathscr{C}} z^\alpha(z - A)^{-1}\, dz$, and so



$$2\pi\delta(A_+{}^{\alpha}) = \int_{\mathscr{C}_1} + \int_{\mathscr{C}_2} + \int_{\mathscr{C}_4} z^{\alpha}(z-A)^{-1}\,\delta(A)(z-A)^{-1}\,dz$$

$$+\,\delta \int_{\mathscr{C}_3} z^{\alpha}(z-A)^{-1}\,dz$$

$$\therefore\ 2\pi\,\|\delta(A_+{}^{\alpha})\| \leqslant 4\pi b^{\alpha-1}\,\|\delta(A)\|$$

$$+\,2\,\|\delta(A)\|\int_a^b y^{\alpha-2}\,dy + \|\delta\|\left\|\int_{\mathscr{C}_3} z^{\alpha}(z-A)^{-1}\,dz\right\|.$$

On letting $b \to \infty$, we find that

$$2\pi\,\|\delta(A_+{}^{\alpha})\| \leqslant 2(1-\alpha)^{-1}\,a^{\alpha-1}\,\|\delta(A)\| + \|\delta\|\left\|\int_{-a}^{a}(iy)^{\alpha}(iy-A)^{-1}\,dy\right\|.$$

Now,

$$\int_{-a}^{a}(iy)^{\alpha}(iy-A)^{-1}\,dy = \int_0^a \{(iy)^{\alpha}(iy-A)^{-1} + (-iy)^{\alpha}(-iy-A)^{-1}\}\,dy$$

$$= \int_0^a y^{\alpha}\{i^{\alpha}(-iy-A) + (-i)^{\alpha}(iy-A)\}(y^2+A^2)^{-1}\,dy$$

$$= \int_0^a y^{\alpha}\{2y\,\sin(\tfrac{1}{2}\pi\alpha) - 2A\,\cos(\tfrac{1}{2}\pi\alpha)\}(y^2+A^2)^{-1}\,dy$$

$$\therefore\ \left\|\int_{-a}^{a}(iy)^{\alpha}(iy-A)^{-1}\,dy\right\|$$

$$\leqslant \pi\alpha\left\|\int_0^a y^{1+\alpha}(y^2+A^2)^{-1}\,dy\right\| + \pi(1-\alpha)\left\|A\int_0^a y^{\alpha}(y^2+A^2)^{-1}\,dy\right\|$$

$$\leqslant \pi\alpha\sup_{\lambda\in\mathbb{R}}\left\{\int_0^a y^{1+\alpha}(y^2+\lambda^2)^{-1}\,dy\right\} + \pi(1-\alpha)\sup_{\lambda\in\mathbb{R}}\left\{|\lambda|\int_0^a y^{\alpha}(y^2+\lambda^2)^{-1}\,dy\right\}.$$

We must now estimate these integrals. If $0 < \lambda < a$, then

$$\int_0^a y^{1+\alpha}(y^2 + \lambda^2)^{-1}\, dy \leqslant \lambda^{-2} \int_0^\lambda y^{1+\alpha}\, dy + \int_\lambda^a y^{\alpha-1}\, dy$$
$$= (2 + \alpha)^{-1}\lambda^\alpha + \alpha^{-1}(a^\alpha - \lambda^\alpha)$$
$$\leqslant \alpha^{-1}a^\alpha,$$

and

$$\lambda \int_0^a y^\alpha(y^2 + \lambda^2)^{-1}\, dy \leqslant \lambda^{-1} \int_0^\lambda y^\alpha\, dy + \lambda \int_\lambda^a y^{\alpha-2}\, dy$$
$$= (\alpha + 1)^{-1}\lambda^\alpha + (1 - \alpha)^{-1}\lambda^\alpha - (1 - \alpha)^{-1}\, \lambda a^{\alpha-1}$$
$$\leqslant 2(1 - \alpha)^{-1}a^\alpha.$$

If $\lambda \geqslant a$, then

$$\int_0^a y^{1+\alpha}(y^2 + \lambda^2)^{-1}\, dy \leqslant \lambda^{-2} \int_0^a y^{1+\alpha} = (2 + \alpha)^{-1}\, \lambda^{-2}a^{2+\alpha}$$
$$\leqslant (2 + \alpha)^{-1}a^\alpha,$$

and

$$\lambda \int_0^a y^\alpha(y^2 + \lambda^2)^{-1}\, dy \leqslant \lambda^{-1} \int_0^a y^\alpha\, dy = (\alpha + 1)^{-1}\, \lambda^{-1}a^{1+\alpha}$$
$$\leqslant (\alpha + 1)^{-1}a^\alpha.$$

Hence

$$\left\| \int_{-a}^a (iy)^\alpha(iy - A)^{-1}\, dy \right\| \leqslant \pi a^\alpha + 2\pi a^\alpha = 3\pi a^\alpha,$$

and so

$$2\pi \, \| \delta(A_+{}^\alpha)\| \leqslant 2(1 - \alpha)^{-1}\, a^{\alpha-1} \, \| \delta(A)\| + 3\pi a^\alpha \, \| \delta \|$$
$$= \{2(1 - \alpha)^{-1} + 3\pi\} \, \| \delta(A)\|^\alpha \, \| \delta \|^{1-\alpha} \qquad (\text{since } a = \| \delta(A)\| \, \| \delta \|^{-1}).$$
$$\therefore \quad \| \delta(A_+{}^\alpha)\| \leqslant 2(1 - \alpha)^{-1} \| \delta(A)\|^\alpha \, \| \delta \|^{1-\alpha}. \quad \blacksquare$$

If we had been content to prove $\| \delta(A_+{}^\alpha)\| \leqslant c(\alpha^{-1} + (1 - \alpha)^{-1}) \, \| \delta(A)\|^\alpha \, \| \delta \|^{1-\alpha}$, then a much simpler estimate of the integral over $\mathscr{C}_3$ could have been made, as was essentially done in [12] for the case $\alpha = \frac{1}{2}$.

4. Smoothness of the Map $S \rightsquigarrow | S |$

Kato [8] proved the following results about the map f_0 defined by $f_0(S) = | S | = (S^*S)^{1/2}$. Here H and H' denote Hilbert spaces.

I. The map $\not{f}_0 : \mathscr{L}(H, H') \to \mathscr{L}(H)$ is almost Lipschitz continuous in the sense that

$$\| \, | S | - | T | \, \| \leqslant 2\pi^{-1} \| S - T \| \{2 + \log((\| S \| + \| T \|) \| S - T \|^{-1})\} \quad (1)$$

for all S and T in $\mathscr{L}(H, H')$.

II. If both H and H' are infinite-dimensional, the map $\not{f}_0$ is not Lipschitz continuous, even when $H' = H$ and $\not{f}_0$ is restricted to the self-adjoint operators.

III. There exists a family $S(t)$, $-1 < t < 1$, of one–one bounded self-adjoint operators in a separable Hilbert space H such that $S(t)$ is a norm-continuously differentiable function of t, and $| S(t) |$ is not weakly differentiable at $t = 0$.

The family $S(t)$ constructed by Kato is not twice differentiable, and he raised the question of whether a family with the properties of III could be constructed which is differentiable to a higher order or even analytic.

It will now be shown that such a family $S(t)$ can be constructed which depends linearly on t.

THEOREM 7. *For every integer $m \geqslant 3$ there exists a family $S_m(t)$ of invertible self-adjoint operators in C^{2m} with the following properties*: (i) *$S_m(t)$ depends linearly on t*; (ii) $\| S_m(t) \| \leqslant 1 + t$; (iii) $\| dS_m(t)/dt \| \leqslant 1$; (iv) $\| [d \, | S_m(t) | /dt]_{t=0} \| \geqslant (8\pi)^{-1} \log(\frac{1}{2}m) - 2$.

Proof. Let $S_m(t) = e^{-m} A_m^{-1} + it\pi^{-1}(A_m B_m - B_m A_m)$, where A_m and B_m are the operators constructed in Theorem 2. It is clear that $S_m(t)$ is a family of self-adjoint operators satisfying (i)–(iii). To verify that each $S_m(t)$ is invertible it suffices to show that it is one–one. Suppose

$$S_m(t) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

i.e.,

$$\begin{bmatrix} U^{-1} & 0 \\ 0 & -U^{-1} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + it \begin{bmatrix} 0 & X \\ -X^* & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $X = e^m \pi^{-1}(UV + VU)$.

$$\therefore \begin{bmatrix} U^{-1}u + itXv \\ -U^{-1}v - itX^*u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix};$$

$$\therefore U^{-1}u + itX(-itUX^*u) = 0;$$

$$\therefore (U^{-1/2}u, U^{-1/2}u) + t^2(U^{1/2}X^*u, U^{1/2}X^*u) = 0.$$

So $u = 0$, and, consequently, $v = 0$ too. Hence $S_m(t)$ is one–one.

Estimate (iv) follows from the formula

$$| S_m(t)| = e^{-m} | A_m |^{-1} + it\pi^{-1}\{(A_mB_m - B_mA_m)U_m + U_m(A_mB_m - B_mA_m) - (| A_m | B_m - B_m | A_m |)\} + O(t^2),$$

where $U_m = A_m | A_m |^{-1}$. The verification of this formula is omitted as it is essentially the same as for Kato's example in [8]. Using Theorem 2 it follows that

$$\|[d | S_m(t)|/dt]_{t=0}\| \geqslant \pi^{-1}(\tfrac{1}{8}\log(\tfrac{1}{2}m) - 2\pi). \quad \blacksquare$$

{It is necessary to take values of m larger than 10^{22} before the right-hand side of (iv) becomes positive!}

THEOREM 8. *There is a family of one–one bounded self-adjoint operators $S(t)$ in a separable Hilbert space H which depends linearly on $t \in \mathbb{R}$, while $| S(t)|$ is not weakly differentiable at $t = 0$.*

Proof. Let $H = \bigoplus \mathbb{C}^{2m}$ and $S(t) = \bigoplus S_m(t)$ where $S_m(t)$ has the properties specified in Theorem 7. Then each $S(t)$ is a one–one bounded self-adjoint operator. Moreover, for each $n > 0$ there is an element $u_n \in H$ such that $|[d(| S(t)| u_n , u_n)/dt]_{t=0} | > n \| u_n \|^2$, so $| S(t)|$ is not weakly differentiable at $t = 0$. $\blacksquare$

Kato's result II mentioned above is also a consequence of Theorem 7. Let us state a C^*-algebra version of I and II.

THEOREM 9. *Let $\mathfrak{a}$ be a C^*-algebra. The map $f_0 : \mathfrak{a} \to \mathfrak{a}$ defined by $f_0(A) = | A |$ is not in general Lipschitz continuous, even when restricted to the self-adjoint elements of $\mathfrak{a}$. However it is almost Lipschitz continuous in the sense that inequality* (1) *holds for all S and $T \in \mathfrak{a}$.*

Proof. Estimate (1) can be derived in the same way as was done by Kato in [8]. $\blacksquare$

5. C^m-SELF-ADJOINT OPERATORS

Chi [3] showed that his conjecture would hold if a conjecture of Kantorovitz on C^m-self-adjoint operators were valid. Consequently, Kantorovitz' conjecture is not valid. To be precise, the following theorem holds.

THEOREM 10. *There is a bounded operator T in a separable Hilbert space which is not C^1-self-adjoint even though $\| \exp(itT)\| \leqslant C(1 + | t |)$ for all $t \in \mathbb{R}$.*

Proof. If A, B and H have the properties specified in Theorem 3, then the operator

$$T = \begin{bmatrix} A & i(AB - BA) \\ 0 & A \end{bmatrix},$$

acting in $H \oplus H$, has the required properties, as is shown in [3]. ∎

For the theory of C^m-self-adjoint operators, see the book [4] of Colojoară and Foiaş, and Kantorovitz, paper [7].

6. Concluding Remarks

Paper [10] contained somewhat weaker versions of Theorems 1, 2, and 3, with $\alpha = 0$, using a more complicated operator W in the result corresponding to Theorem 1. The operator W used here was suggested by Kahan [5]. The results of [10] have been employed to give negative answers to other questions in [11], although it would have been better to have used Theorems 1 and 2 above. See, in particular, the remark at the end of [11].

The question left open in [10] is still unanswered. We conclude with a related question for derivations. If $A = A^* \in \mathscr{D}(\delta)$, and $\delta(\delta(A)) = 0$, is $|A| \in \mathscr{D}(\delta)$? More generally, is $\|\delta(|A|)\|$ bounded by a suitable function of $\|\delta(A)\|$ and $\|\delta(\delta(A))\|$?

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